A path integral approach to random motion with nonlinear friction

2010 J. Phys. A: Math. Theor. 43 025003

(http://iopscience.iop.org/1751-8121/43/2/025003)

J. Phys. A: Math. Theor. **43** (2010) 025003 (12pp) doi:10.1088/1751-8113/43/2/025003

A path integral approach to random motion with nonlinear friction

A Baule[1], E G D Cohen[1] and H Touchette[2]

[1] The Rockefeller University, 1230 York Avenue, New York, NY 10065, USA
[2] School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK

Received 30 October 2009, in final form 11 November 2009
Published 10 December 2009
Online at stacks.iop.org/JPhysA/43/025003

Abstract
Using a path integral approach, we derive an analytical solution of a nonlinear and singular Langevin equation, which has been introduced previously by P-G de Gennes as a simple phenomenological model for the *stick-slip* motion of a solid object on a vibrating horizontal surface. We show that the optimal (or most probable) paths of this model can be divided into two classes of paths, which correspond physically to a sliding or slip motion, where the object moves with a non-zero velocity over the underlying surface, and a stick-slip motion, where the object is stuck to the surface for a finite time. These two kinds of basic motions underlie the behavior of many more complicated systems with solid/solid friction and appear naturally in de Gennes' model in the path-integral framework.

PACS numbers: 02.50.−r, 05.40.−a, 46.55.+d, 46.65.+g

1. Introduction

We study an old but still only very partially understood problem: the dynamics of a solid object moving over a solid surface. In practice this is a very complicated and as yet unsolved problem, although there is a wealth of experiments, since the general problem is very old and ubiquitous in nature, ranging from geology and engineering to physics and biology. The basic difficulty lies in the very complex nature and behavior of the solid/solid interface, which leads to a complicated stick-slip motion of the object [1].

Following P-G de Gennes, we study in detail one of the simplest phenomenological models, far from those of most practical interest, but as a starting point to develop a new theoretical approach to describe basic aspects of the above-mentioned problem. Ignoring all details of the solid/solid interfacial layer, de Gennes considered a simple Langevin equation for the velocity $v(t)$ of a solid object of mass m on a horizontal vibrating surface [2, 3]:

$$m\dot{v}(t) + \alpha v(t) + \sigma[v(t)]\Delta_F = \xi(t). \tag{1}$$

In this equation two kinds of friction between the object and the surface, over which it moves, appear: (a) a *dynamic* friction (sometimes called kinetic friction), which is taken proportional to v as in the Stokes friction term in fluids and characterized by the dynamical friction coefficient α; (b) a *static* friction (sometimes called dry friction), which is given by the $\sigma(v)\Delta_F$ term. Here, the function $\sigma(v)$ is the sign function of the object's velocity v, i.e. $\sigma(v) = +1, 0, -1$ for $v > 0, = 0, < 0$, respectively, and Δ_F is the coefficient (strength) of the static friction. The $\sigma(v)\Delta_F$ term represents a nonlinearity and, in fact, a singularity in the Langevin equation (1), since $\sigma(v)$ is discontinuous at $v = 0$. Physically, this term ensures that the solid object is subject to a static friction, which is equal to Δ_F and acts always, via $\sigma(v)$, opposite to the direction of motion of the object. Both friction coefficients α and Δ_F are here assumed to be constant, which implies that the object moves over an isotropic surface.

The contribution of de Gennes in [2] is to consider the motion of the object driven by externally applied one-dimensional vibrations of the underlying surface. These vibrations are represented by an external noise $\xi(t)$, which has the properties of Gaussian white noise:

$$\langle \xi(t) \rangle = 0, \tag{2}$$

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t'), \tag{3}$$

with noise strength D. The complicated solid/solid interface is therefore replaced by a static and a dynamic friction term, and randomness is externally induced by Gaussian noise.

In this paper we use a path integral approach to study the properties of the nonlinear de Gennes' model equation (1). While de Gennes has used a Fokker–Planck approach to obtain approximate results for the transition probability (defined in the next section), the path integral approach provides a more dynamical picture of the statistical properties of the object, on the basis of the most probable or *optimal* paths in the velocity–time plane. Using these optimal paths, we obtain an analytical solution for the transition probability in the saddle-point approximation for small values of D. As one of our main results we show that the optimal paths of equation (1) can be divided into two classes of paths, which correspond physically to *slip* motion, where the object moves with a velocity $v \neq 0$ over the underlying surface, and *stick-slip* motion, where the object is stuck to the surface with $v = 0$ for a finite time. The existence of these two kinds of motions is a basic element of almost all dynamical systems with solid/solid friction and appear naturally in de Gennes' model in the path integral framework.

In the following we present a detailed account of the path-integral approach to nonlinear stochastic systems. We analyze the structure of the optimal paths of equation (1), and derive an analytical expression for the transition probability, defined in the next section.

2. Path integral approach

The transition probability or propagator $f(v, t|v_0, t_0)$ gives the probability of finding the object with velocity v at time t, given that it had a velocity v_0 at the initial time t_0. Using very many paths generated by the Gaussian white noise $\xi(t)$, the transition probability can be obtained empirically from many realizations of the de Gennes equation (1) for fixed initial and final conditions. In the asymptotic time limit $t \to \infty$ the transition probability converges to a stationary distribution $p(v)$, which can be derived from equation (1). So, introducing an effective potential

$$U(v) = \frac{v^2}{2\tau_m} + |v|\Delta, \tag{4}$$

with the characteristic inertial time $\tau_m = m/\alpha$ and $\Delta \equiv \Delta_F/m$, equation (1) takes the form of Brownian motion in the nonlinear potential $U(v)$:

$$\dot{v}(t) = -U'(v) + \xi(t)/m. \tag{5}$$

Due to the confining character of $U(v)$ a stationary distribution of the velocity coordinate exists and can be calculated from equation (5) using standard methods [4]. The result is

$$p(v) = N\,\mathrm{e}^{-\gamma U(v)}, \tag{6}$$

where $\gamma \equiv m^2/D$ and N is a normalization constant. Clearly, the stationary distribution $p(v)$ is symmetric under a change of sign of v. In fact, also the propagator $f(v, t|v_0, t_0)$ has to be symmetric under the change $v_0 \to -v_0$ and $v \to -v$. This forward/backward symmetry is physically due to the fact that the surface is assumed to be isotropic and the applied noise is symmetric, so that no bias in forward or backward direction is induced. As a consequence, all the statistical properties of the velocity also have this forward/backward symmetry.

In order to obtain the transition probability we use a path integral approach, in which the transition probability $f(v, t|v_0, t_0)$ is formally expressed as an integral over all paths leading from the initial state (v_0, t_0) to the final state (v, t). Historically, the path integral was introduced into quantum mechanics by Feynman [5], and later on into statistical mechanics by Kac [6] and Onsager and Machlup [7, 8]. The path integral approach was generalized to nonlinear systems in [9] and to fluctuations in a class of nonequilibrium systems in [10]. For more background information we refer to the recent survey [11].

For the dynamics of equation (1), the path integral is given by [5]

$$f(v, t|v_0, t_0) = \int_{(v_0,t_0)}^{(v,t)} J[v]\,\mathrm{e}^{-\gamma A[\dot{v},v]}\mathcal{D}v, \tag{7}$$

where $A[\dot{v}, v]$ is a functional of $v(s)$,

$$A[\dot{v}, v] = \int_{t_0}^{t} \mathcal{L}(\dot{v}(s), v(s))\,\mathrm{d}s, \tag{8}$$

which is usually referred to as the *action* associated with the path $v(s)$. Here, $\mathcal{L}$ is an Onsager–Machlup Lagrangian, in analogy with mechanics, given by

$$\mathcal{L}(\dot{v}, v) = \tfrac{1}{4}(\dot{v} + U'(v))^2. \tag{9}$$

In equation (7) the integral $\int \mathcal{D}v$ denotes an integral over all paths $v(s)$ from (v_0, t_0) to (v, t). The Jacobian $J[v]$ originates from the transformation $\xi(t) \to v(t)$ and is a functional of $v(s)$ due to the nonlinearity of the force $-U'(v)$ in equation (5) [12, 13]:

$$J[v] = \mathrm{e}^{\frac{1}{2}\int_{t_0}^{t} U''(v(s))\mathrm{d}s}. \tag{10}$$

We evaluate the path integral equation (7) in the saddle-point approximation, which proceeds as follows (cf [14]). For large γ the dominant contribution to the path integral is due to a particular path $v^*(s)$ that maximizes the exponent in equation (7), or equivalently, which minimizes the action $A[\dot{v}, v]$:

$$\delta A[\dot{v}^*, v^*] = 0. \tag{11}$$

This condition yields an Euler–Lagrange (EL) equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{v}^*} - \frac{\partial \mathcal{L}}{\partial v^*} = 0, \tag{12}$$

for the path $v^*(s)$, which is the path with highest probability among all paths connecting (v_0, t_0) and (v, t), i.e. it is the most probable or *optimal* path. We can then expand the action $A[\dot{v}, v]$ in the neighborhood of the optimal path using

$$v(s) = v^*(s) + z(s), \tag{13}$$

where $z(s)$ is the deviation from the optimal path. Clearly, the boundary conditions for $z(s)$ are $z(t_0) = z(t) = 0$. Expanding the action around $v^*(s)$ yields

$$A[\dot{v}, v] = A[\dot{v}^*, v^*] + \int_{t_0}^{t} \mathrm{d}s \left.\frac{\delta A}{\delta v(s)}\right|_{v^*} z(s) + \frac{1}{2}\int_{t_0}^{t} \mathrm{d}s \int_{t_0}^{t} \mathrm{d}s' \left.\frac{\delta^2 A}{\delta v(s)\delta v(s')}\right|_{v^*} z(s)z(s') + \cdots. \tag{14}$$

Here, the linear term vanishes due to equation (11) and, using equations (8) and (9), the second-order term can be calculated as

$$\frac{1}{2}\int_{t_0}^{t} \mathrm{d}s \int_{t_0}^{t} \mathrm{d}s' \left.\frac{\delta^2 A}{\delta v(s)\delta v(s')}\right|_{v^*} z(s)z(s') = \int_{t_0}^{t} \mathrm{d}s[\dot{z}(s)^2 + \Omega(v^*(s))z(s)^2], \tag{15}$$

with

$$\Omega(v) \equiv U''(v)^2 + U'(v)U'''(v). \tag{16}$$

The leading orders in the expansion of the action are thus

$$A[\dot{v}, v] = A[\dot{v}^*, v^*] + \int_{t_0}^{t} \mathrm{d}s[\dot{z}(s)^2 + \Omega(v^*(s))z(s)^2]. \tag{17}$$

Substituting only the zeroth-order term of this expansion into the path integral equation (7) then yields the saddle-point approximation of the transition probability $f(v, t|v_0, t_0)$:

$$f(v, t|v_0, t_0) \cong J[v^*]\,\mathrm{e}^{-\gamma A[\dot{v}^*, v^*]}, \tag{18}$$

valid for large γ. Keeping, in addition, the second-order term in equation (17) yields the corrected form

$$f(v, t|v_0, t_0) \cong J[v^*]\,\mathrm{e}^{-\gamma A[\dot{v}^*, v^*]} F[v^*], \tag{19}$$

where the fluctuation factor $F[v^*]$ is determined by the path integral

$$F[v^*] = \int_{(0,t_0)}^{(0,t)} \mathrm{e}^{-\gamma \int_{t_0}^{t} \mathrm{d}s[\dot{z}(s)^2 + \Omega(v^*(s))z(s)^2]} \mathcal{D}z. \tag{20}$$

The analytic calculation of the fluctuation factor for the nonlinear potential equation (4) is beyond the scope of this paper. In the following we focus on the properties of the optimal paths and consider the transition probability in the saddle-point approximation equation (18).

3. Solution of the Euler–Lagrange (EL) equation

For the Lagrangian, equation (9), the EL-equation assumes the explicit form

$$\ddot{v}^* - \frac{v^*}{\tau_m^2} - \sigma(v^*)\frac{\Delta}{\tau_m} = 0. \tag{21}$$

A complete picture of the properties of the optimal paths and, on the basis of equation (18), of the transition probability $f(v, t|v_0, t_0)$ is obtained by solving equation (21) under the given boundary conditions, which are fixed initial and final velocities (v_0, t_0) and (v, t), respectively. Equation (21) consists in fact of two different equations, namely one for positive v^*, in which case $\sigma(v^*) = +1$ and one for negative v^*, where $\sigma(v^*) = -1$. Each of these two equations is straightforward to solve if the boundary conditions are such that v^* is always positive or negative, i.e. if $v^*(s)$ remains entirely on either the upper ($v > 0$) or the lower half ($v < 0$) of the velocity–time (v–s) plane. In that case one finds the solutions (dropping the $*$ for optimal path in the following)

$$v_\pm(s) = B_\pm\,\mathrm{e}^{s/\tau_m} + C_\pm\,\mathrm{e}^{-s/\tau_m} \mp \Delta\tau_m, \tag{22}$$

where + refers to the upper half plane and − to the lower, respectively. On the upper half plane, the prefactors B_+ and C_+ are determined by

$$v_+(t_0) = v_0, \qquad v_+(t) = v, \tag{23}$$

which leads to

$$B_+ = \frac{\mathrm{e}^{t/\tau_m}(v + \Delta\tau_m) - \mathrm{e}^{t_0/\tau_m}(v_0 + \Delta\tau_m)}{\mathrm{e}^{2t/\tau_m} - \mathrm{e}^{2t_0/\tau_m}}, \tag{24}$$

$$C_+ = \frac{\mathrm{e}^{t/\tau_m}(v_0 + \Delta\tau_m) - \mathrm{e}^{t_0/\tau_m}(v + \Delta\tau_m)}{\mathrm{e}^{(t-t_0)/\tau_m} - \mathrm{e}^{-(t-t_0)/\tau_m}}. \tag{25}$$

The prefactors B_- and C_-, for the lower half plane, are then obtained by simply changing $\Delta \to -\Delta$ in equations (24) and (25).

The forward/backward symmetry of the velocity statistics (discussed below equation (6)) implies a symmetry between the upper and lower half of the v–s plane: the basic solution $v_+(s)$ between (v_0, t_0) and (v, t) on the upper half plane is just the mirror image of the path $v_-(s)$ between $(-v_0, t_0)$ and $(-v, t)$ on the lower half plane. Therefore, in the following discussion it is sufficient to consider only a positive initial velocity v_0, without loss of generality. Due to this symmetry, the action for the paths $v_+(s)$ and $v_-(s)$ is the same and can be calculated by substituting equation (22) into equations (9) and (8). This yields for the basic action

$$A[\dot{v}_\pm, v_\pm] = \int_{t_0}^{t} \mathcal{L}(\dot{v}_\pm, v_\pm)\,\mathrm{d}s = \Lambda(v, t; v_0, t_0), \tag{26}$$

where we define

$$\Lambda(v, t; v_0, t_0) \equiv \frac{(\mathrm{e}^{t/\tau_m}(\Delta\tau_m + |v|) - \mathrm{e}^{t_0/\tau_m}(\Delta\tau_m + v_0))^2}{2\tau_m(\mathrm{e}^{2t/\tau_m} - \mathrm{e}^{2t_0/\tau_m})}. \tag{27}$$

In addition to the set of basic solutions of equation (22), another formal solution of the EL-equation (21) is given by $v(s) = 0$. Using the formal solutions $v_\pm(s)$ and $v(s) = 0$ one can construct the full solution of the EL-equation for fixed initial and final points as linear combinations of the three formal solutions. Two distinct classes of solutions then arise, namely *indirect* paths that partly follow the $v = 0$ axis and *direct* paths that do not. They are both discussed in the following.

3.1. Direct paths

Direct paths are characterized by continuous $v(s)$ and $\dot{v}(s)$. They remain either entirely on one half of the v–s plane or cross the $v = 0$ axis. The former are given by the solutions $v_\pm(s)$, while paths that cross the $v = 0$ axis consist of one branch on the upper half plane and one on the lower half plane (cf figures 1(*a*) and (*b*)). The direct paths (indicated by the subscript d) that remain on the upper plane are simply parametrized by

$$v_d(s)\big|_{v_0,t_0}^{v,t} = v_+(s)\big|_{v_0,t_0}^{v,t}, \tag{28}$$

where $v_+(s)$ is given by equation (22) and the boundary conditions are indicated.

For the direct crossing paths we have to consider that the upper branch is given by $v_+(s)$ under the boundary conditions (v_0, t_0) and $(0, \bar{t})$, where $\bar{t}$ is the time at which the path crosses the $v = 0$ axis, and the lower branch is given by $v_-(s)$ under the boundary conditions $(0, \bar{t})$ and (v, t). Direct crossing paths (indicated by the subscript d and the superscript $\times$) are thus parametrized by

$$v_d^\times(s) = \begin{cases} v_+(s)\big|_{v_0,t_0}^{0,\bar{t}}, & t_0 \leqslant s \leqslant \bar{t} \\ v_-(s)\big|_{0,\bar{t}}^{v,t}, & \bar{t} < s \leqslant t. \end{cases} \tag{29}$$

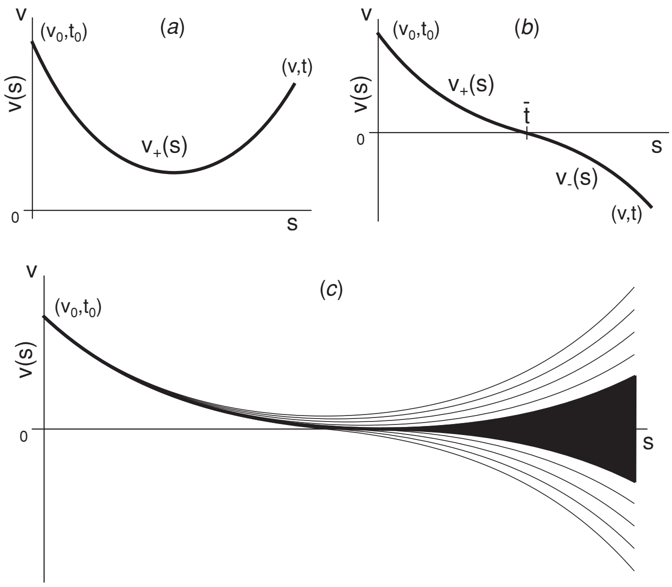

Figure 1. Direct paths in the v–s plane. (a) A direct path on the upper half plane, parametrized by equation (28). (b) A direct crossing path, parametrized by equation (29). (c) Here, we plot a number of direct paths that are very close together initially. When a direct path crosses the $v = 0$ axis a jump from a positive to a negative curvature occurs, while direct paths that remain on the upper half plane continue with a positive curvature. The jump in the curvature leads to a forbidden region (black region).

Here, the crossing time $\bar{t}$ is always $t_0 < \bar{t} < t$ and is determined from the condition of a continuous acceleration at the crossover point:

$$\dot{v}_+(\bar{t}) = \dot{v}_-(\bar{t}). \tag{30}$$

Using equation (22), this condition leads to a fourth-order equation for $\bar{t}$:

$$\begin{aligned} 0 = \mu^4 \Delta\tau_m &- \mu^3[(\Delta\tau_m + v_0)\,\mathrm{e}^{t_0/\tau_m} + (\Delta\tau_m - v)\,\mathrm{e}^{t/\tau_m}] \\ &+ \mu[(\Delta\tau_m + v_0)\,\mathrm{e}^{(t_0+2t)/\tau_m} + (\Delta\tau_m - v)\,\mathrm{e}^{(2t_0+t)/\tau_m}] \\ &- \Delta\tau_m\,\mathrm{e}^{(2t_0+2t)/\tau_m}, \end{aligned} \tag{31}$$

where $\mu \equiv \mathrm{e}^{\bar{t}/\tau_m}$. Equation (31) has a unique real root $t_0 < \bar{t} < t$.

The action associated with the direct paths that remain on the upper half plane is

$$A[\dot{v}_d, v_d] = A[\dot{v}_+, v_+] = \Lambda(v, t; v_0, t_0), \tag{32}$$

which follows immediately from equations (28) and (26). The action of the direct crossing paths on the other hand, consists of contributions from the upper and the lower branch, i.e.

$$\begin{aligned} A\big[\dot{v}_d^\times, v_d^\times\big] &= \int_{t_0}^{\bar{t}} \mathcal{L}(\dot{v}_+, v_+)\,\mathrm{d}s + \int_{\bar{t}}^{t} \mathcal{L}(\dot{v}_-, v_-)\,\mathrm{d}s \\ &= \Lambda(0, \bar{t}; v_0, t_0) + \Lambda(v, t; 0, \bar{t}), \end{aligned} \tag{33}$$

using equations (29) and (26). Throughout this paper, the function Λ of equation (27) is adapted to the case at hand, by replacing v, t; v_0, t_0 by the appropriate velocities and times.

A crucial observation is then that not all initial and final points in the entire velocity–time plane can be connected by a direct path (cf figure 1(c)). This is due to a jump in the curvature

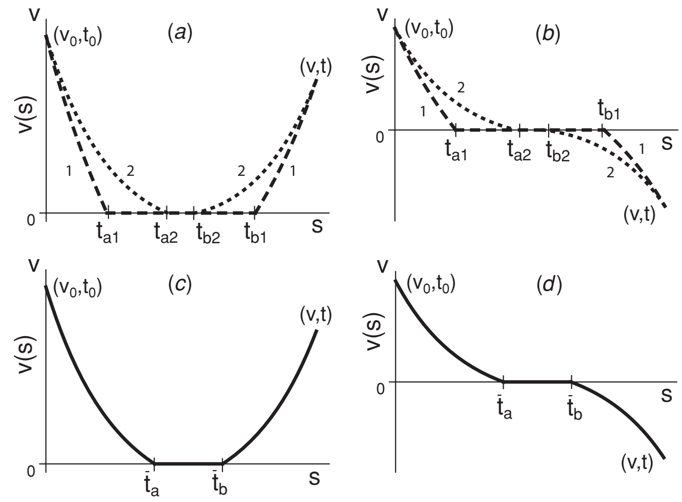

Figure 2. Indirect paths in the v–s plane. (*a*) Two examples of indirect paths on the upper half plane, parametrized by equation (34). Path 1 (long dashed curve) crosses the $v = 0$ axis at the times t_{a1} and t_{b1}. Path 2 (short dashed curve) at the times t_{a2} and t_{b2}. (*b*) Two indirect paths that cross the $v = 0$ axis, parametrized by equation (34). (*c*) and (*d*) The unique indirect paths with minimal action between given initial and final points, parametrized by equation (41).

of the direct path, when the $v = 0$ axis is crossed: the upper branch $v_+(s)$ is always convex with $\ddot{v}_+ \approx \Delta$ close to the $v = 0$ axis, while the lower branch $v_-(s)$ is always concave with $\ddot{v}_- \approx -\Delta$ close to the $v = 0$ axis (cf equation (21)). There exists thus a forbidden region in the v–s plane, i.e. a region that cannot be reached by any direct path from a given initial point. As $t - t_0$ becomes larger this region grows exponentially, so that eventually, as $t \to \infty$, fewer and fewer final points can be reached by a direct optimal path. However, this seems to contradict the fact that a stationary distribution exists and has to be approached by $f(v, t|v_0, t_0)$ in the asymptotic time limit. Therefore, direct optimal paths cannot represent the full solution of the EL-equation (21). The key is to consider other solutions that satisfy equation (21) piecewise. This allows us to construct another class of solutions, namely indirect paths.

3.2. Indirect paths

Indirect paths consist of three parts: a relaxation branch from the initial point (v_0, t_0) to the axis at $(0, t_a)$, a part along the zero axis from $(0, t_a)$ to $(0, t_b)$ and an excitation branch from $(0, t_b)$ to the final point (v, t). The relaxation branch is given by $v_+(s)$ under the boundary conditions (v_0, t_0) and $(0, t_a)$ and the excitation branch either by $v_+(s)$ or by $v_-(s)$ under the boundary conditions $(0, t_b)$ and (v, t), respectively (cf figures 2(*a*) and (*b*)). Indirect paths (indicated by the subscript id) are thus parametrized by

$$v_{id}(s) = \begin{cases} v_+(s)|_{v_0,t_0}^{0,t_a}, & t_0 \leqslant s \leqslant t_a \\ 0, & t_a < s < t_b \\ v_\pm(s)|_{0,t_b}^{v,t}, & t_b \leqslant s \leqslant t. \end{cases} \tag{34}$$

Clearly, the times t_a, t_b have to satisfy the conditions $t_0 < t_a \leqslant t_b < t$. All three parts of $v_{id}(s)$ satisfy the EL-equation piecewise. We note that indirect paths are valid solutions of equation (21) because the boundary conditions of the EL-equation are the fixed initial and final points of the optimal path. If one specifies instead the initial velocity and the initial acceleration of the object, indirect paths cannot arise.

The action associated with the indirect paths parametrized by equation (34) is

$$\begin{aligned} A[\dot{v}_{id}, v_{id}] &= \int_{t_0}^{t_a} \mathcal{L}(\dot{v}_+, v_+)\, \mathrm{d}s + \int_{t_b}^{t} \mathcal{L}(\dot{v}_\pm, v_\pm)\, \mathrm{d}s \\ &= \Lambda(0, t_a; v_0, t_0) + \Lambda(v, t; 0, t_b), \end{aligned} \tag{35}$$

which follows from equations (34) and (26). Since t_a and t_b are not specified, there are in principle infinitely many indirect paths between (v_0, t_0) and (v, t) possible, which all have different actions according to equation (35) (cf figures 2(*a*) and (*b*)). But, among all these indirect paths there exists a unique indirect path with minimal action. This 'optimal' indirect path is obtained by determining the minimum of $A[\dot{v}_{id}, v_{id}]$ of equation (35), with respect to t_a and t_b. This is a minimization problem subject to the inequality constraint

$$t_a \leqslant t_b. \tag{36}$$

In order to find the solution for this minimization we note that

$$\frac{\partial}{\partial t_a} A[\dot{v}_{id}, v_{id}] = \frac{\partial}{\partial t_a} \Lambda(0, t_a; v_0, t_0) \tag{37}$$

is independent of t_b and has only one zero for $t_a \in [t_0, t]$, at which the axis is crossed with a positive slope. Likewise

$$\frac{\partial}{\partial t_b} A[\dot{v}_{id}, v_{id}] = \frac{\partial}{\partial t_b} \Lambda(v, t; 0, t_b), \tag{38}$$

is independent of t_a. These properties imply that $A[\dot{v}_{id}, v_{id}]$ of equation (35) has a unique global minimum as a function of t_a and t_b and is monotonically increasing away from it. Due to this monotonicity, the minimum of $A[\dot{v}_{id}, v_{id}]$, subject to the inequality constraint $t_a \leqslant t_b$, is either given by the global unconstrained minimum or, if this minimum cannot be attained because it violates the inequality constraint equation (36), by a minimum of $A[\dot{v}_{id}, v_{id}]$ subject to the equality constraint $t_a = t_b$.

We then obtain the following for the minimum of $A[\dot{v}_{id}, v_{id}]$.

(1) First, the unconstrained minimum is determined by setting each of the time derivatives, equations (37) and (38), equal to zero. This yields the times

$$\bar{t}_a \equiv t_0 + \tau_m \ln\left(1 + \frac{v_0}{\Delta \tau_m}\right), \tag{39}$$

$$\bar{t}_b \equiv t - \tau_m \ln\left(1 + \frac{|v|}{\Delta \tau_m}\right). \tag{40}$$

We note that the time $\bar{t}_a$ is just the time at which a noise-free path, described by equation (5) with $\xi(t) = 0$, would relax to the $v = 0$ axis starting from (v_0, t_0). Likewise, $\bar{t}_b$ is the time at which a noise-free path starting at (v, t) would reach the axis, moving backward in time. We parametrize the optimal indirect path specified by $\bar{t}_a$ and $\bar{t}_b$ by (cf figures 2(*c*) and (*d*))

$$\bar{v}_{id}(s) = \begin{cases} v_+(s)|^{0,\bar{t}_a}_{v_0,t_0}, & t_0 \leqslant s \leqslant \bar{t}_a \\ 0, & \bar{t}_a < s < \bar{t}_b \\ v_\pm(s)|^{v,t}_{0,\bar{t}_b}, & \bar{t}_b \leqslant s \leqslant t. \end{cases} \tag{41}$$

The associated action is then given by equation (35), where t_a and t_b are replaced by $\bar{t}_a$ and $\bar{t}_b$, respectively:

$$\begin{aligned} A[\dot{\bar{v}}_{id}, \bar{v}_{id}] &= \Lambda(0, \bar{t}_a; v_0, t_0) + \Lambda(v, t; 0, \bar{t}_b) \\ &= \Lambda(v, t; 0, \bar{t}_b) \\ &= U(v). \end{aligned} \tag{42}$$

The term $\Lambda(0, \bar{t}_a; v_0, t_0)$ on the right-hand side of the first line of equation (42) vanishes, because it is the action associated with the noise-free relaxation branch, for which the Lagrangian is identically zero (cf equation (9)). The third line follows upon substituting equation (40) into equation (27).

(2) Second, if the global minimum does not exist, i.e. if $\bar{t}_b < \bar{t}_a$, one has to determine the minimum of equation (35) under the equality constraint $t_a = t_b$. For this, one has to solve

$$\frac{\partial}{\partial t_a}[\Lambda(0, t_a; v_0, t_0) + \Lambda(v, t; 0, t_a)] = 0 \tag{43}$$

for t_a. One finds that there is a unique real solution of this equation $\in [t_0, t]$, which is identical with the time $\bar{t}$ determined by equation (31). The associated action is then given by equation (35) with $t_a = t_b = \bar{t}$, i.e. $\Lambda(0, \bar{t}; v_0, t_0) + \Lambda(v, t; 0, \bar{t})$, which is just equal to the action of the direct crossing paths, $A\left[\dot{v}_d^{\times}, v_d^{\times}\right]$ of equation (33).

3.3. Optimal paths in the v–s plane

Having determined the two classes of solutions of the EL-equation (21), one can now find the unique optimal path between given initial and final points, direct or indirect, using the minimal action principle. We thus compare the actions of the direct and indirect paths separately for the paths on the upper half plane and for the crossing paths.

First, we consider the paths on the upper half plane: the direct paths are given by equation (28) with the associated action $A[\dot{v}_d, v_d]$, equation (32). The indirect paths are specified by equation (41) with the associated action $A[\dot{\bar{v}}_{id}, \bar{v}_{id}]$, equation (42). From the condition

$$A[\dot{v}_d, v_d] = A[\dot{\bar{v}}_{id}, \bar{v}_{id}], \tag{44}$$

which, using equations (32) and (42), is equivalent to

$$U(v) = \Lambda(v, t; v_0, t_0), \tag{45}$$

one can derive a critical value of v, denoted by $u^+(t)$, such that $A[\dot{\bar{v}}_{id}, \bar{v}_{id}] \leqslant A[\dot{v}_d, v_d]$ if $v \leqslant u^+(t)$. Equation (45) leads to a quadratic equation for v, which has the relevant root

$$u^+(t) \equiv \mathrm{e}^{(t-t_0)/\tau_m}(\Delta\tau_m + v_0) - \Delta\tau_m - \sqrt{\left(v_0^2 + 2\Delta\tau_m v_0\right)\left(\mathrm{e}^{2(t-t_0)/\tau_m} - 1\right)}. \tag{46}$$

This means that if $v < u^+(t)$ and $t \geqslant \bar{t}_a$, the action of the indirect path is lower than that of the direct path. The condition $t \geqslant \bar{t}_a$ is necessary for indirect paths to exist.

Second, we consider crossing paths. The direct crossing paths $v_d^{\times}(s)$ are given by equation (29) with the associated action $A[\dot{v}_d^{\times}, v_d^{\times}]$, equation (33). As before, the indirect paths are specified by equation (41) with the associated action $A[\dot{\bar{v}}_{id}, \bar{v}_{id}]$, equation (42). The results of the minimization of equation (35) show the following: when $\bar{t}_b > \bar{t}_a$ the indirect crossing path always has a lower action than the direct crossing path between the same initial and final points, i.e. $A[\dot{\bar{v}}_{id}, \bar{v}_{id}] \leqslant A[\dot{v}_d^{\times}, v_d^{\times}]$. On the other hand, when $\bar{t}_b < \bar{t}_a$, the indirect paths of equation (41) no longer exist and the direct crossing path then has the lowest action. Therefore, from the condition

$$\bar{t}_b = \bar{t}_a, \tag{47}$$

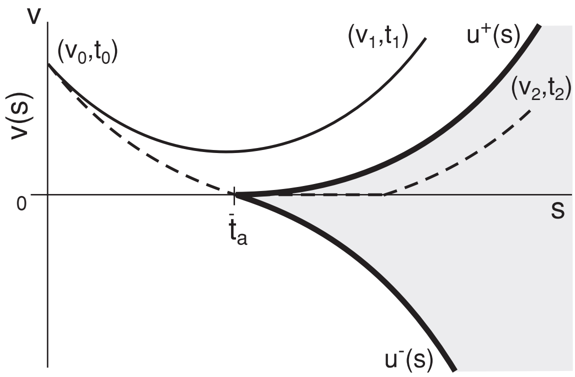

Figure 3. Diagram of the optimal paths in a velocity-time plane for a fixed initial point (v_0, t_0) and varying final points. The bold black solid curves are given by $u^+(s)$, equation (46), on the upper half plane, and by $u^-(s)$, equation (48), on the lower half plane. When the final point of the optimal path lies outside of the shaded region, as here (v_1, t_1), the optimal path is a direct path (solid line). Otherwise, if the final point lies inside the shaded region, as here (v_2, t_2), the optimal path is an indirect path (dashed line).

one can derive a critical value of v on the lower half plane, namely

$$u^-(t) \equiv -\Delta\tau_m \left(\frac{\Delta\tau_m}{\Delta\tau_m + v_0} \mathrm{e}^{(t-t_0)/\tau_m} - 1 \right), \tag{48}$$

so that the action of the indirect path is lower than that of the direct path, if $v \geqslant u^-(t)$ and $t \geqslant \bar{t}_a$.

It follows from this discussion that for a given initial point (v_0, t_0) the optimal path is an indirect path if the end point (v, t) lies in the interval $u^-(t) \leqslant v \leqslant u^+(t)$ with $t \geqslant \bar{t}_a$, otherwise the optimal path is a direct path (cf figure 3).

The curves $u^+(t)$ and $u^-(t)$ represent boundaries in the v–s plane, separating different qualitative dynamical behaviors of the moving object in terms of direct and indirect optimal paths. In fact, physically, direct paths can be considered as representing a pure *slip* motion of the object, where $v \neq 0$ apart from one crossing point. Indirect paths follow the $v = 0$ axis for a finite time and thus represent physically a *stick-slip* motion.

4. Transition probability

From the structure of the optimal paths in the v–s plane, the transition probability $f(v, t|v_0, t_0)$ follows directly via the saddle-point approximation equation (18). The Jacobian as given by equation (10) reads explicitly for the potential $U(v)$ of equation (4):

$$J[v^*] = \mathrm{e}^{(t-t_0)/(2\tau_m)+\Delta\int_{t_0}^{t}\delta(v^*(s))\mathrm{d}s}. \tag{49}$$

Here, the term $\int_{t_0}^{t}\delta(v^*(s))\,\mathrm{d}s$ is different from zero only when the optimal path $v^*(s)$ crosses the $v = 0$ axis. This means that for the direct paths on the upper half plane the Jacobian is just a function of t_0, t, which can effectively be absorbed into the normalization of the transition probability. Only for the direct crossing paths and the indirect paths the Jacobian contributes significantly. For the direct crossing paths we obtain

$$\int_{t_0}^{t} \delta\left(v_d^{\times}(s)\right) \mathrm{d}s = \frac{1}{\left|\dot{v}_d^{\times}(\bar{t})\right|}, \tag{50}$$

and for the indirect paths

$$\int_{t_0}^{t} \delta\left(\bar{v}_{id}(s)\right) \mathrm{d}s = \frac{1}{2|\dot{\bar{v}}_{id}(\bar{t}_a)|} + \frac{1}{2|\dot{\bar{v}}_{id}(\bar{t}_b)|} = \frac{1}{\Delta}. \tag{51}$$

In deriving these expressions we have used the representation of the delta function [15]

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i), \tag{52}$$

where the x_i are the zeros of $g(x)$. In the last step of equation (51) we have substituted $\bar{t}_a$ and $\bar{t}_b$ from equations (39) and (40) into the time derivative of $\bar{v}_{id}$, equation (41).

With these results for the Jacobian, we can express the transition probability of equation (18), for a given initial point (v_0, t_0), as follows.

For $t \leqslant \bar{t}_a$ (defined in equation (39)), no indirect optimal paths occur, so that (a) for a final velocity in the interval $-\infty < v \leqslant 0$ only the direct crossing paths contribute to the action in the expression for the transition probability, equation (18); (b) for $0 < v < \infty$ only direct paths on the upper plane contribute. Using equations (32) and (33), respectively, in equation (18), and considering the contribution of the Jacobian, equation (49) with equation (50), we obtain the transition probability

$$f(v, t|v_0, t_0) = \mathcal{N}_1 \begin{cases} \mathrm{e}^{-\gamma[\Lambda(0,\bar{t};v_0,t_0)+\Lambda(v,t;0,\bar{t})]+\Delta/|\dot{v}_d^{\times}(\bar{t})|}, & -\infty < v \leqslant 0 \\ \mathrm{e}^{-\gamma\Lambda(v,t;v_0,t_0)}, & 0 < v < \infty, \end{cases} \tag{53}$$

where $\mathcal{N}_1$ is a normalization factor.

For $t > \bar{t}_a$ indirect paths appear and the structure of the optimal paths (as discussed in section 3.3) indicates that (a) for a final velocity in the interval $-\infty < v \leqslant u^-(t)$ the direct crossing paths contribute to the action in equation (18); (b) for $u^-(t) < 0 < u^+(t)$ the indirect paths contribute; (c) for $u^+(t) \leqslant v < \infty$ the direct paths on the upper plane contribute. Using equations (33), (35) and (32), respectively, in equation (18), and considering the contribution of the Jacobian, equation (49) with equations (50) and (51), respectively, the transition probability reads

$$f(v, t|v_0, t_0) = \mathcal{N}_2 \begin{cases} \mathrm{e}^{-\gamma[\Lambda(0,\bar{t};v_0,t_0)+\Lambda(v,t;0,\bar{t})]+\Delta/|\dot{v}_d^{\times}(\bar{t})|}, & -\infty < v \leqslant u^- \\ \mathrm{e}^{-\gamma U(v)+1}, & u^- < v < u^+ \\ \mathrm{e}^{-\gamma\Lambda(v,t;v_0,t_0)}, & u^+ \leqslant v < \infty, \end{cases} \tag{54}$$

where $\mathcal{N}_2$ is a normalization factor.

Since both $u^- \to -\infty$ and $u^+ \to \infty$ as $t \to \infty$, only indirect paths contribute to the transition probability in the asymptotic time limit and we recover the stationary distribution equation (6) in this limit from equation (54):

$$\lim_{t\to\infty} f(v, t|v_0, t_0) = p(v). \tag{55}$$

From the transition probability $f(v, t|v_0, t_0)$ one can construct joint probability distributions for arbitrary sequences of n-points in the velocity–time plane. Due to the Markovian character of equation (1), the joint probability of finding the object at n successive points $(v_0, t_0) \to (v_1, t_1) \to (v_2, t_2) \to \cdots \to (v_n, t_n)$ in the velocity–time plane is just given by the product of n transition probabilities $f(v_n, t_n|v_{n-1}, t_{n-1})$.

5. Conclusion

We have derived a characterization of the optimal paths of the de Gennes' equation (1) within the path integral framework. The optimal paths can be divided into two classes.

(a) *Direct* optimal paths, with continuous $v(s)$ and $\dot{v}(s)$, which physically can be considered as representing a *pure slip* motion of the object. (b) *Indirect* optimal paths, with continuous $v(s)$ and discontinuous $\dot{v}(s)$, which follow partly the $v = 0$ axis and represent physically a *stick-slip* motion. We have shown that for a given initial point (v_0, t_0) the optimal path will *either* be a direct *or* an indirect path depending on the location of the final point (v, t) in the velocity–time plane (cf figure 3).

This analysis of the optimal paths leads to an analytical result for the transition probability $f(v, t|v_0, t_0)$ in the saddle-point approximation. The calculation of correction terms to this result, such as the fluctuation factor $F(v^*)$, equation (20), which takes into account the second-order term in the expansion of the action, equation (14), is left for future work. However, we emphasize that higher order corrections leave the properties and the structure of the optimal paths, and therefore also their slip and stick-slip character, unchanged.

From a physical point of view, the friction terms in equation (1) represent a very simple phenomenological model of the solid/solid interaction between the object and the surface. By studying generalizations of equation (1), incorporating, e.g. two-dimensional or memory effects, one could model more complicated surface properties, such as surface anisotropies or defects. A comparison of such more realistic, but still phenomenological models, with experiments will lead to a better understanding of the effects of surface inhomogeneities on the motion of a solid object on a vibrating surface.

References

[1] Persson B N J 2000 *Sliding Friction: Physical Principles and Applications* (Berlin: Springer)
[2] de Gennes P G 2005 *J. Stat. Phys.* **119** 953
[3] Buguin A, Brochard F and de Gennes P-G 2006 *Eur. Phys. J.* E **19** 31
[4] Risken H 1996 *The Fokker–Planck Equation: Methods of Solution and Applications* (Berlin: Springer)
[5] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[6] Kac M 1949 *Trans. Am. Math. Soc.* **65** 1
[7] Onsager L and Machlup S 1953 *Phys. Rev.* **91** 1505
[8] Machlup S and Onsager L 1953 *Phys. Rev.* **91** 1512
[9] Freidlin M I and Wentzell A D 1984 *Random Perturbations of Dynamical Systems* (Berlin: Springer)
[10] Taniguchi T and Cohen E G D 2007 *J. Stat. Phys.* **126** 1
[11] Majumdar S N 2005 *Curr. Sci.* **89** 2076
[12] Graham R 1973 *Springer Tracts in Modern Physics* vol 66 (New York: Springer) p 1
[13] Hunt K L C and Ross J 1981 *J. Chem. Phys.* **75** 976
[14] Kleinert H 2009 *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (Singapore: World Scientific)
[15] Arfken G B and Weber H J 2005 *Mathematical Methods for Physicists* (New York: Academic)